# Chapter 9

# Zweier I-Convergent Double Sequence Spaces Defined by a Modulus Function

## 9.1 Introduction

An *Orlicz function* is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.(see[4,47]). If the convexity of the regular function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called as *Modulus function*. This function was introduced by Nakano[58]. Ruckle[64] and Maddox[56] further investigated the modulus function with applications to sequence spaces.

In this chapter we introduce the following class of sequence spaces:

$$_2\mathcal{Z}^I(f) = \{(x_{ij}) \in {}_2\omega : I - \lim f(|x'_{ij} - L|) = 0, \text{ for some } L \in \mathbb{C} \},$$

$$_2\mathcal{Z}_0^I(f) = \{(x_{ij}) \in {}_2\omega : I - \lim f(|x'_{ij}|) = 0\},$$

$$_2\mathcal{Z}_\infty^I(f) = \{(x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} :$$

$$\text{there exist } K > 0 : f(|x'_{ij}|) \geq K \in I\}.$$

$$_2\mathcal{Z}_\infty(M) = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} f(|x'_{ij}|) < \infty\}$$

Throughout we denote

$$m^I_{_2\mathcal{Z}}(f) = {}_2\mathcal{Z}^I_\infty(f) \cap {}_2\mathcal{Z}(f) \quad and \quad m^I_{_2\mathcal{Z}_0}(f) = {}_2\mathcal{Z}^I_\infty(f) \cap {}_2\mathcal{Z}_0(f).$$

Throughout the article, for the sake of convenience we will denote by $Z^p(x_{ij}) = x'$, $Z^p(y_{ij}) = y'$, $Z^p(z_{ij}) = z'$ for $x, y, z \in \omega$.

"Under the leadership of our dear masters Banach and Steinhauss we were practicing in Lwów intricacies of mathematics"- Orlicz-1968.

## 9.2 Main Results

Theorem 9.2.1. For any modulus function $f$, the classes of sequences ${}_2\mathcal{Z}^I(f),\ {}_2\mathcal{Z}_0^I(f), m^I_{{}_2\mathcal{Z}}(f)$ and $m^I_{{}_2\mathcal{Z}_0}(f)$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(f)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I(f)$ and let $\alpha, \beta$ be scalars. Then

$$I - \lim f(|x'_{ij} - L_1|) = 0,\ \text{for some} L_1 \in \mathbb{C}\ ;$$

$$I - \lim f(|y'_{ij} - L_2|) = 0,\ \text{for some} L_2 \in \mathbb{C}\ ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x'_{ij} - L_1|) > \frac{\epsilon}{2}\} \in I, \qquad [9.1]$$

$$A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|y'_{ij} - L_2|) > \frac{\epsilon}{2}\} \in I. \qquad [9.2]$$

Since $f$ is a modulus function, we have

$$\begin{aligned} f(|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)) &\leq f(|\alpha||x'_{ij} - L_1|) + f(|\beta||y'_{ij} - L_2|) \\ &\leq f(|x'_{ij} - L_1|)\ +\ f(|y'_{ij} - L_2|) \end{aligned}$$

Now, by [9.1] and [9.2],

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|(\alpha x'_{ij} + \beta y'_{ij}) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I(f)$. Hence ${}_2\mathcal{Z}^I(f)$ is a linear space.

We state the following result without proof in view of Theorem 2.1.

Theorem 9.2.2. The spaces $m^I_{2\mathcal{Z}}(f)$ and $m^I_{2\mathcal{Z}_0}(f)$ are normed linear spaces, normed by

$$||x^{'}_{ij}||_{*} = \sup_{i,j} f(|x^{'}_{ij}|). \qquad [9.3]$$

Theorem 9.2.3. A sequence $x = (x_{ij}) \in m^I_{2\mathcal{Z}}(f)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{k} - x^{'}_{N_\epsilon}|) < \epsilon\} \in \ m^I_{2\mathcal{Z}}(f) \qquad [9.4]$$

Proof. Suppose that $L = I - \lim x^{'}$. Then

$$B_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{'}_{ij} - L| < \frac{\epsilon}{2}\} \in \ m^I_{2\mathcal{Z}}(f). For\ all\ \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x^{'}_{N_\epsilon} - x^{'}_{ij}| \leq |x^{'}_{N_\epsilon} - L| + |L - x^{'}_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i,j) \in B_\epsilon$. Hence

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{ij} - x^{'}_{N_\epsilon}|) < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

Conversely, suppose that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x^{'}_{ij} - x^{'}_{N_\epsilon}|) < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

That is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{'}_{ij} - x^{'}_{N_\epsilon}| < \epsilon\} \in m^I_{2\mathcal{Z}}(f)$$

for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x^{'}_{ij} \in [x^{'}_{N_\epsilon} - \epsilon, x^{'}_{N_\epsilon} + \epsilon]\} \in m^I_{2\mathcal{Z}}(f)\ for\ all\ \epsilon > 0.$$

Let $J_\epsilon = [x^{'}_{N_\epsilon} - \epsilon, x^{'}_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{2\mathcal{Z}}(f)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{2\mathcal{Z}}(f)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{2\mathcal{Z}}(f)$. This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij}^{'} \in J\} \in m_{2\mathcal{Z}}^{I}(f)$$

that is

$$diam J \leq diam J_{\epsilon}$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_{\epsilon} = I_0 \supseteq I_1 \supseteq ..... \supseteq I_{ij} \supseteq ..........$$

with the property that $diam\ I_{ij} \ \leq \ \frac{1}{2}\ diam\ I_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij}^{'} \in I_{ij}\} \in m_{2\mathcal{Z}}^{I}(f)$ for (k=1,2,3,4,......).

Then there exists a $\xi \in \cap I_k$ where $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $\xi = I - \lim x^{'}$. So that $f(\xi) = I - \lim f(x^{'})$, that is $L = I - \lim f(x^{'})$.

Theorem 9.2.4. Let $f$ and $g$ be modulus functions that satisfy the $\triangle_2$-condition. If $X$ is any of the spaces ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}_0^I, m_{2\mathcal{Z}}^I$ and $m_{2\mathcal{Z}_0}^I$ , then the following assertions hold

(a)$X(g) \subseteq X(f.g)$,

(b)$X(f) \cap X(g) \subseteq X(f+g)$

Proof. (a) Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(g)$. Then

$$I - \lim_{ij} g(|x_{ij}^{'}|) = 0 \qquad [9.5]$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 < t < \delta$. Write $y_{ij} = g(|x_{ij}^{'}|)$ and consider

$$\lim_{i,j} f(y_{ij}) = \lim_{i,j} f(y_k)_{y_{ij}<\delta} + \lim_{i,j} f(y_{ij})_{y_{ij}>\delta}$$

We have

$$\lim_{i,j} f(y_{ij}) \leq f(2) \lim_{i,j}(y_{ij}) \qquad [9.6]$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Since $f$ is non-decreasing, it follows that

$$f(y_{ij}) < f(1 + \frac{y_{ij}}{\delta}) < \frac{1}{2}f(2) + \frac{1}{2}f(\frac{2y_k}{\delta})$$

Since $f$ satisfies the $\triangle_2$-condition, we have

$$f(y_{ij}) < \frac{1}{2}K\frac{y_{ij}}{\delta}f(2) + \frac{1}{2}K\frac{y_{ij}}{\delta}f(2) = K\frac{y_{ij}}{\delta}f(2)$$

Hence

$$\lim_{i,j} f(y_{ij}) \leq max(1, K)\delta^{-1}f(2)\lim_{i,j}(y_{ij}). \qquad [9.7]$$

From [9.5], [9.6] and [9.7], we have $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f.g)$. Thus ${}_2\mathcal{Z}_0^I(g) \subseteq {}_2\mathcal{Z}_0^I(f.g)$. The other cases can be established following similar technique.

(b) Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f) \cap {}_2\mathcal{Z}_0^I(g)$. Then $I - \lim_{i,j} f(|x_{ij}^{'}|) = 0$ and $I - \lim_{i,j} g(|x_{ij}^{'}|) = 0$

The rest of the proof follows from the following equality

$$\lim_{i,j}(f + g)(|x_{ij}^{'}|) = \lim_{ij} f(|x_{ij}^{'}|) + \lim_{i,j} g(|x_{ij}^{'}|).$$

Corollary 9.2.5. $X \subseteq X(f)$ for $X = {}_2\mathcal{Z}^I, {}_2\mathcal{Z}_0^I, m_{2\mathcal{Z}}^I$ and $m_{2\mathcal{Z}_0}^I$ .

Theorem 9.2.6. The spaces ${}_2\mathcal{Z}_0^I(f)$ and $m_{2\mathcal{Z}_0}^I(f)$ are solid and monotone.

Proof. We shall prove the result for the sequence space ${}_2\mathcal{Z}_0^I(f)$. Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f)$. Then

$$I - \lim_{i,j} f(|x_{ij}^{'}|) = 0. \qquad [9.8]$$

Let $(\alpha_{ij})$ be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [9.8] and the following inequality

$$f(|\alpha_{ij}x'_{ij}|) \leq |\alpha_{ij}|f(|x'_{ij}|) \leq f(|x'_{ij}|) \; for \; all \; (i,j) \in \mathbb{N} \times \mathbb{N}.$$

That the space ${}_2\mathcal{Z}_0^I(f)$ is monotone follows from the Lemma 1.12. For $m^I_{{}_2\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 9.2.7. The spaces ${}_2\mathcal{Z}^I(f)$ and $m^I_{{}_2\mathcal{Z}}(f)$ are neither solid nor monotone in general .

Proof. We prove this result by providing a counter example. Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of $X$ defined as follows

Let $(x_{ij}) \in X$ and let $(y_{ij}) \in X_K$ be such that

$$(y_{ij}) = \begin{cases} (x_{ij}) & \text{if i+j is even,} \\ 0, & otherwise. \end{cases}$$

Consider the sequence $(x_{ij})$ defined by $(x_{ij}) = 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$ but its K-stepspace preimage does not belong to ${}_2\mathcal{Z}^I(f)$. Thus ${}_2\mathcal{Z}^I(f)$ is not monotone. Hence ${}_2\mathcal{Z}^I(f)$ is not solid.

Theorem 9.2.8. The spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are sequence algebras.

Proof. We prove that the sequence space ${}_2\mathcal{Z}_0^I(f)$ is a sequence algebra. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I(f)$. Then

$$I - \lim f(|x'_{ij}|) = 0 \quad and \quad I - \lim f(|y'_{ij}|) = 0$$

Then we have

$$I - \lim f(|x'_{ij}.y'_{ij}|) = 0$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I(f)$ is a sequence algebra. For the space ${}_2\mathcal{Z}_0^I(f)$, the result can be proved similarly.

Theorem 9.2.9. The spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not convergence free in general.

Proof. We give a counter example to prove this result.

Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence $(x_{ij})$ and $(y_{ij})$ defined by

$$x_{ij} = \frac{1}{i+j} \quad and \;\; y_{ij} = i + j \;\text{ for all } (i, j) \in \mathbb{N} \times \mathbb{N}.$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$. Hence the spaces ${}_2\mathcal{Z}_0^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not convergence free.

Theorem 9.2.10. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not symmetric.

Proof. Let $A \in I$ be infinite and $f(x) = x$ for all $x \in [0, \infty)$. If

$$x_{ij} = \begin{cases} 1, & \text{for } (i, j) \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.14 $(x_{ij}) \in {}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f)$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(m)\pi(n)} \notin {}_2\mathcal{Z}^I(f)$ and $x_{\pi(m)\pi(n)} \notin {}_2\mathcal{Z}_0^I(f)$. Hence ${}_2\mathcal{Z}^I(f)$ and ${}_2\mathcal{Z}_0^I(f)$ are not symmetric.

Theorem 9.2.11. Let $f$ be a modulus function. Then ${}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f) \subset {}_2\mathcal{Z}_\infty^I(f)$.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I(f)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f(|x'_{ij} - L|) = 0$$

We have $f(|x'_{ij}|) \leq f(|x'_{ij} - L|) + f(|L|)$. Taking the supremum over $(i, j)$ on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I(f)$. The inclusion ${}_2\mathcal{Z}_0^I(f) \subset {}_2\mathcal{Z}^I(f)$ is obvious.

Theorem 9.2.12. The function $\hbar : m^I_{2\mathcal{Z}}(f) \rightarrow \mathbb{R}$ is the Lipschitz function, where $m^I_{2\mathcal{Z}}(f) = {}_2\mathcal{Z}_\infty^I(f) \cap {}_2\mathcal{Z}^I(f)$, and hence uniformly continuous.

Proof. Let $x, y \in m^I{}_{2\mathcal{Z}}(f)$, $x \neq y$. Then the sets

$$A_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < ||x - y||_*\} \in m^I_{2\mathcal{Z}}(f),$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_k - \hbar(y)| < ||x - y||_*\} \in m^I_{2\mathcal{Z}}(f).$$

Hence also $B = B_x \cap B_y \in m^I_{2\mathcal{Z}}(f)$, so that $B \neq \Phi$. A Now taking $(i, j)$ in $B$,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y_{ij} - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m^I_{2\mathcal{Z}_0}(f)$ the result can be proved similarly.

Theorem 9.2.13. If $x, y \in m^I_{2\mathcal{Z}}(f)$, then $(x.y) \in m^I_{2\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \epsilon\} \in m^I_{2\mathcal{Z}}(f),$$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \epsilon\} \in m^I_{2\mathcal{Z}}(f).$$

Now,

$$\begin{aligned} |x_{ij}y_{ij} - \hbar(x)\hbar(y)| &= |x_{ij}y_{ij} - x_{ij}\hbar(y) + x_{ij}\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_{ij}||y_{ij} - \hbar(y)| + |\hbar(y)||x_{ij} - \hbar(x)| \qquad [9.9] \end{aligned}$$

As $m^I_{2\mathcal{Z}}(f) \subseteq {}_2\mathcal{Z}^I_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|x_{ij}| < M$ and $|\hbar(y)| < M$.

Using eqn[9.9] we get

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon \; = \; 2M\epsilon$$

for all $(i,j) \in B_x \cap B_y \in m^I(f)$. Hence $(x.y) \in m^I_{2\mathcal{Z}}(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m^I_{2\mathcal{Z}_0}(f)$ the result can be proved similarly.